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Matrix Factorization: A Comparative Study of the Performance of its Fundamental Methods and an Investigation of the Conditions for Obtaining Orthonormal Matrices with Rational Elements

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ABSTRACT

This research conducts a study on matrices and matrix analysis, outlining the essential fundamental definitions required for this purpose.

We provide a historical review of the majority of scholars and researchers who have studied and analyzed matrices, documenting each researcher's methodology. Furthermore, we detail the advantages and disadvantages of each method as applied in previous periods. A comparison was also conducted between QR decomposition and LU decomposition. Finally, we established the necessary conditions for obtaining normalized orthogonal matrices with rational elements.

KEYWORDS: Matrix, Matrix Factorization, QR Factorization, LU Factorization.

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1. Introduction

The history of matrifactorization represents anintellectual journey extending from ancient civilizations to the modern era. Matrix factorization is considered a ,significant branch of mathematics due to its wide range of applications in a diversity of fields such as physics, engineering and computer science.

The Beginnings: Historical evidence indicates that the roots of matrix factorization date back to ancient civilizations such as China, Egypt, and Babylon. In the third century BCE, "Seki Takakazu" was among the first mathematicians to show interest in the study of matrices, making important contributions to solving systems of linear equations through matrix methods [14].

The Middle Ages: During the Middle Ages, Arab and Islamic mathematicians continued the development of matrix-related concepts. In the eleventh century, "Al-Khwarizmi" emerged as a pioneer in algebra, equations using matrices. "Ibn al-Haytham" also contributed to the development of determinants, which later became an essential mathematical tool in matrix factorization [14].

The Seventeenth Century: With the advent of the seventeenth century, matrix factorization witnessed a qualitative leap, thanks to the contributions of European mathematicians. Specifically in 1693, "Gottfried Leibniz" independently introduced concepts related to matrices through his work on determinants, paving the way for the later development of matrix theory as an independent mathematical discipline [14].

The Eighteenth Century:

The eighteenth century saw significant progress in matrix studies. Gabriel Cramer emerged as a pioneer for developing rules related to matrix arithmetic and systems of equations.

. "Carl Friedrich Gauss" also made significant contributions to the theory of determinants, which helped enhance the understanding and analysis of matrices [14].

The Nineteenth Century: In the nineteenth century, the concept of "Linear Algebra" became more clearly established, leading to the development of new mathematical tools for matrix factorization. In 1848, "James Joseph Sylvester" coined the term "matrix" to describe an ordered set of elements [4].

The Twentieth Century: The twentieth century witnessed a revolution in matrix factorization, thanks to the contributions of prominent mathematicians such as "Arthur Cayley" and "David Hilbert." Cayley established the foundations of modern matrix theory, while Hilbert focused on developing new methods for solving large systems of linear equations [8].

The Modern Era: With tremendous technological advancements, new applications for matrix factorization have emerged in various fields such as physics, engineering, and computer science. In the field of artificial intelligence, matrices are widely used in machine learning and natural language processing applications [14].

2. Research Objective

This research aims to identify the principal methodsof matrix factorization, examine the advantages and disadvantages of each method, and provide a comparative analysis between them

3. Research Significance

The significance of this research lies in its comprehensive study of matrices and all their factorization methods, serving as a preliminary step towards the development of new approaches that are simpler, more efficient, and less computationally complex.

4. Discussion and Results

4.1. Methods in Matrix Factorization

4.1.1. CARL FRIEDRICH GAUSS (1777–1855) [1]

He was a German mathematician, statistician, physicist, and scientist.

Contributions:

- He introduced the Gauss-Jordan algorithm for solving linear systems, which relies on the factorization of a matrix into a product of two elementary matrices.
- He introduced the concept of "reduction to row echelon form" for matrices, which is used to solve linear systems.

Example:

Consider the following system of linear equations:

$$2x + 3y = -4$$
$$4x - y = 6$$

The coefficient matrix for this system is:

$$\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$$

And the augmented matrix is:

$$\left(\begin{array}{cc|c} 2 & 3 & -4 \\ 4 & -1 & 6 \end{array}\right)$$

Using the Gauss-Jordan method, the augmented matrix can be transformed into reduced row echelon form:

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

This implies that the solution to the system is: $x = 1, y = -2$

Advantages of the Gauss-Jordan method:

- **Ease of Use:** The elementary matrix simplifies the process of transforming the matrix into a reduced form.
- **Accuracy of Results:** The Gauss-Jordan method, with the elementary matrix, ensures the procurement of accurate solutions for the equations.
- **Versatility:** The Gauss-Jordan method can be applied to solve systems of linear equations of all orders.

4.1.2. WILHELM JORDAN (1842–1929) [2]

Wilhelm Jordan (in German) was a German mathematician and university professor, born in Württemberg. He was also a member of the German National Academy of Sciences Leopoldina.

Contributions:

- He developed Jordan's algorithm for solving linear systems, which relies on the factorization of a matrix into a product of two elementary matrices. It is noted that an elementary matrix is an invertible square matrix that results from a single elementary row operation on the identity matrix.

- He introduced the concept of the "Jordan normal form" of matrices, which is used in determining matrix properties such as invertibility and diagonalizability.

Disadvantages of Jordan's algorithm: This algorithm may become computationally inefficient when applied to large linear systems, as it requires a considerable number of computational operations.

4.1.3. FELIX KLEIN (1849–1925) [3]

He was a German mathematician.

Contributions:

- He introduced the concept of a "skew-symmetric matrix," which plays a significant role in the factorization of matrices into a product of two matrices.
- A skew-symmetric matrix is a square matrix A that satisfies: $A^T = -A$.

4.1.4. JAMES JOSEPH SYLVESTER (1814–1897) [5], [6]

An English mathematician credited with coining the term "matrix" in 1848, who made significant contributions to matrix theory.

- He developed the theory of determinants, which is used to calculate the value of a matrix's determinant function.
- Sylvester first introduced the concept of the symmetric matrix in a research paper titled "On the Theory of Matrices."
- He defined a symmetric matrix as a matrix that is equal to its transpose.
- He elucidated the properties of symmetric matrices, such as the fact that they always have real eigenvalues.
- Sylvester continued to develop his theory on symmetric matrices in another research paper titled "On the Theory of Symmetric Matrices."
- He introduced the concept of the "partitioned symmetric matrix" and the "compound symmetric matrix."
- He discussed the applications of symmetric matrices in fields such as analytic geometry and physics.

Disadvantages of determinant theory: The computation of a matrix determinant can be difficult, especially for large matrices.

4.1.5. DAVID HILBERT (1862–1943) [7]

He is a German mathematician of the first rank among the mathematicians of the twentieth century. Instead of delivering a lecture in 1900, Hilbert preferred to present before 250 participating mathematicians at the International Congress of Mathematicians a list of complex problems comprising 23 mathematical problems that would foster research in various aspects of mathematics.

Contributions:

He developed the theory of integral matrices, which is used to study the properties of matrices of infinite ranks.

An integral matrix is a square matrix whose elements are integrals of integrable functions.

For example:

$$\begin{pmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx \\ \int_a^b h(x)\,dx & \int_a^b k(x)\,dx \end{pmatrix}$$

The theory of integral matrices aims to understand the relationship between the properties of the integral matrix and the properties of the functions from which its elements are composed.
Some applications in the theory of integral matrices include the following:

- **Analysis:** Finding other matrices that can be multiplied together to yield a specific integral matrix.
- **Similarity:** Determining if two integral matrices are similar. We say that two matrices A, B are similar if an invertible matrix R can be found such that $B = RAR^{-1}$ or $A = R^{-1}BR$.
- **Eigenvalues:** Finding the eigenvalues of an integral matrix.
- **Applications:** Using integral matrices to solve integral equations, differential equations, signal processing, and more.
- **In Numerical Analysis:** Integral matrices are used to solve integral equations and differential equations by approximate methods.
- **In Signal Processing:** Integral matrices are used to filter and analyze signals.
- **In Physics:** Integral matrices are used to model complex physical systems, such as quantum mechanics systems.
- **Note:** The theory of integral matrices is a relatively advanced mathematical field.

4.1.6. ARTHUR CAYLEY (1821–1898) [8]

An English mathematician regarded as one of the founders of Linear Algebra and matrix theory, Arthur Cayley introduced matrices in 1855 as a representation of linear elements. He also studied orthogonal matrices and defined them as square $(n \times n)$ matrices satisfying the following conditions:

1. The columns of the matrix are mutually orthogonal. Meaning that the inner product of any two columns of the matrix is equal to zero.
2. The length of each column of the matrix is equal to 1. Meaning that the inner product of any column of the matrix with itself yields 1.

Cayley established the following two theorems:

- **Theorem 1:** Let S be a skew-symmetric matrix over $\mathbb{R}^{n\times n}$ with no eigenvalue equal to -1. Then the matrix $Q = (S - I)^{-1}(S + I)$ is an orthogonal matrix.
- **Theorem 2:** Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Then $S = (Q - I)^{-1}(Q + I)$ is a skew-symmetric matrix.

4.1.7. GIVENS ROTATIONS (1950) [9], [10]

Givens Rotation:

Householder transformations are effective for zeroing out elements of a matrix; for instance, the first Householder transformation converts all elements of the first vector (the first column) to zero, except for the first element. Givens rotations, however, zero out matrix elements more selectively. A Givens rotation is a specific rotation matrix used to introduce a zero into a targeted element of a matrix.

A **Givens matrix** is obtained from the identity matrix by replacing specific elements with elements derived from a particular angle of rotation.
A Givens matrix can be represented as follows:

$$G(i,k,\theta)=\begin{bmatrix}1 & \cdots & 0 & \cdots & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & & \vdots & & \vdots\\ 0 & \cdots & c & \cdots & s & \cdots & 0\\ \vdots & & \vdots & \ddots & \vdots & & \vdots\\ 0 & \cdots & -s & \cdots & c & \cdots & 0\\ \vdots & & \vdots & & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1\end{bmatrix}\begin{matrix}\\ \\ \leftarrow i\\ \\ \leftarrow k\\ \\ \\ \end{matrix}$$

where $c=\cos(\theta)$ and $s=\sin(\theta)$ are placed in rows and columns i and k.
It is evident that a Givens rotation is orthogonal, and that pre-multiplication by $G(i,k,\theta)^T$ performs a counter-clockwise rotation by an angle θ in the (i,k) coordinate plane.

Procedure:

To zero out the element a_{ij} in matrix A using a Givens rotation, we perform the following steps:

1. Determine the angle θ such that: $c=\cos(\theta)=\frac{a_{ii}}{\sqrt{a_{ii}^2+a_{ij}^2}}$ $s=\sin(\theta)=\frac{a_{ij}}{\sqrt{a_{ii}^2+a_{ij}^2}}$
2. Apply the Givens matrix $G(i,k,\theta)$ to matrix A to adjust the targeted elements, progressively modifying it towards the desired form.

Example: Using Givens Rotations to Zero an Element in a Matrix (Corrected Formulation)

Let us consider the following matrix A:

$$A=\begin{pmatrix}3 & 2 & 1\\ -1 & 1 & 2\\ 3 & 1 & 4\end{pmatrix}$$

Our objective is to use Givens rotations to transform matrix A into an upper triangular matrix by progressively zeroing out the sub-diagonal elements. We will begin by zeroing out the element a_{32} (which has a value of 1).

Step 1: Determine the Rotation Matrix $G(2,3,\theta)$

To zero out element a_{32}, we need a rotation matrix that acts on the second and third rows. We determine the values of $c=\cos(\theta)$ and $s=\sin(\theta)$ using the elements of the second column in these two rows, which are $a_{22}=1$ and $a_{32}=1$.

1. Calculate r:

$$r=\sqrt{a_{22}^2+a_{32}^2}=\sqrt{1^2+1^2}=\sqrt{2}$$

2. Calculate c and s:

$$c=\frac{a_{22}}{r}=\frac{1}{\sqrt{2}},s=\frac{a_{32}}{r}=\frac{1}{\sqrt{2}}$$

Construct the Givens matrix $G(2,3,\theta)$ that acts on the second and third rows:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Step 2: Apply the Rotation to Matrix A

We pre-multiply matrix A by the rotation matrix G. This rotation will update the second and third rows and zero out the target element a_{32}.

$$A' = G \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ -1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix}$$

Result:

The new matrix A' after zeroing out the element a_{32} is:

$$A' = \begin{pmatrix} 3 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & 0 & \sqrt{2} \end{pmatrix}$$

We observe that the element at position (3,2) is now zero, as required.

Next Steps:

We would continue to apply additional Givens rotations to zero out the remaining sub-diagonal elements (such as a_{21} and a_{31}) until an upper triangular matrix is obtained.

4.1.8. Gram-Schmidt [11]

The history of Gram-Schmidt research dates back to 1935, when both Jørgen Gram and Erhard Schmidt presented a mathematical method to transform a set of vectors into an orthonormal set.

Jørgen Gram:

Danish mathematician (1850–1916).

He introduced the concept of the "Gram process" to calculate the angle between two vectors in a Hilbert space. Gram died before publishing his research on the Gram process.

Erhard Schmidt:

German mathematician (1876–1959).

He developed the "Schmidt process" to transform a set of vectors into an orthonormal set.

Gram and Schmidt's research was published together in 1935.

The Gram-Schmidt method is used to transform a matrix into the product of an orthogonal matrix and an upper triangular matrix.

The Gram-Schmidt process begins by processing the matrix for $k = 1$ to n:

$$u_k = v_k - \sum_{j=1}^{k-1} (v_k \cdot q_j) q_j$$

$$q_k = \frac{u_k}{\|u_k\|}$$

As for the Modified Gram-Schmidt method, if $(b_1, b_2, \ldots, b_n)$ is a set of vectors that forms a basis, we then build an orthonormal basis $(u_1, u_2, \ldots, u_n)$ as follows: according to [15]

$$\tilde{u}_j = b_j - \sum_{k=1}^{j-1} \frac{(\tilde{u}_k^T b_k)}{(\tilde{u}_k^T \tilde{u}_k)} \tilde{u}_k$$

$$u_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|}$$

Benefits of the Modified Gram-Schmidt Method:

- **Avoidance of Error Accumulation:** The Modified Gram-Schmidt method helps to reduce the accumulation of numerical errors that can occur in the original method.
- **Improved Numerical Stability:** The Modified Gram-Schmidt method is more numerically stable than the original method, particularly when the input vectors are close to each other.
- **Ease of Implementation:** The Modified Gram-Schmidt method is easier to understand and implement than the classical approach.

Drawbacks of the Gram-Schmidt Method in Matrix Factorization:

- If the values of the input data are close to each other, the Gram-Schmidt method may produce inaccurate or unexpected results.
- The Gram-Schmidt algorithm has relatively high complexity, especially for large matrices, because it requires performing numerous computational operations, such as dot products and subtractions.
- As the matrix size increases, both the computational time and memory requirements increase significantly. The Gram-Schmidt method may lead to a loss of accuracy because it uses the new vectors step-by-step, which can lead to the accumulation of rounding errors.
- In general, the Gram-Schmidt method is not the optimal choice for matrix factorization in all cases.

4.1.9. John H. Staib [12]:

In 1969, Staib raised several objections to the Gram-Schmidt process, considering it tedious inconvenient. Consequently, he proposed an alternative approach known as matrix method.

4.1.10. Anthony E. Hoffman [13]:

In 1970, Hoffman compared the Gram-Schmidt method with Staib's alternative method and concluded that the Gram-Schmidt method remained reasonably effective despite its drawbacks.

4.1.11. Householder Method in QR Factorization (1958) [15]:

The American mathematician Marvin Householder introduced the Householder method, commonly referred to as Householder reflections.

Let A be an $n \times n$ matrix. The method involves constructing a series of orthogonal matrices H_i such that:

$$H = \prod_{i=1}^{n-1} H_i$$

and it reduces and transforms A into upper triangular form. Calculating H_i is a simple generalization of calculating H_1.
We name each column using vectors:

$$A = [u_1, u_2, \dots, u_n]$$

We define $v = \|u_1\|e_1 - u_1$, where $e_1^T = [1,0,\dots,0]$.
The Householder matrix is then called H_1, a matrix using the first column:

$$H_1 = I - \frac{2vv^T}{v^T v}$$

Then H_1 is orthogonal and $H_1 A$ is a matrix in which the first column is $\|u_1\|e_1$. Then we continue with the submatrix obtained from $H_1 A$ by deleting the first row and first column until we get:

$$H_{n-1}H_{n-2} \dots H_1 A = R$$
$$A = H^{-1}R = QR$$

Where:
- Q is an orthogonal matrix.
- R is an upper triangular matrix.

This factorization is used in numerous applications, including:
1. Solving eigenvalue problems.
2. Matrix transformations.
3. Solving linear systems.
4. Data processing.

Drawbacks of the Householder method:
- The computations involved can be relatively complex; however, with the evolution of technology, complex calculations have become easier and faster to execute.
- **Inefficiency in some cases:** Alternative methods exist for factoring large or dense matrices, such as methods based on Givens rotations, which may be more efficient, particularly for sparse matrices.

The steps for factoring a matrix with the row-reduced form and Gaussian elimination are arduous, as are the Gram-Schmidt process and Householder transformations.

4.2. LU Factorization [16]

The origin of LU factorization dates back to the early twentieth century, when several mathematicians such as Lievin Rimann and Ernest Croutch paved the way for LU factorization. However, the analysis was not fully formulated and generalized until the 1940s, when it was introduced by mathematicians Hans Kohn and William Feller independently.
- **Definition (1) Orthogonal Matrix:** A matrix A is said to be orthogonal if it satisfies the condition $A^T = A^{-1}$.
- **Definition (2) Upper Triangular Matrix:** A matrix A is called an upper triangular matrix if all elements below the main diagonal are equal to zero.

- **Definition (3) Lower Triangular Matrix:** A matrix A is called a lower triangular matrix if all elements above the main diagonal are equal to zero.
- **Definition (4) LU Factorization:** Also known as lower-upper factorization, it is a factorization used to transform a matrix into a product of an upper triangular matrix and a lower triangular matrix.
- **Definition (5) QR Factorization:** It is a factorization used to transform a matrix into a product of an orthogonal matrix and an upper triangular matrix.
- **Definition (6) Sparse Matrix:** A matrix is called sparse if most of its elements are zeros.

4.2.1. The Difference Between QR Factorization and LU Factorization

QR factorization and LU factorization are two common techniques for matrix factorization. Each method possesses specific advantages and disadvantages that make it suitable for particular computational applications.

Table 1: Comparison between LU Factorization and QR Factorization

Feature	QR Factorization	LU Factorization
Purpose	Transformation of a matrix into a product of an orthogonal matrix and an upper triangular matrix.	Transformation of a matrix into a product of an upper triangular matrix and a lower triangular matrix.
Applications	Solving systems of linear equations, data classification, signal processing.	Solving systems of linear equations, finding the inverse of a matrix.
Properties	Numerically stable, does not require pivoting.	Faster in some cases, not numerically stable.
When to Choose	Numerical stability is important, the matrix is sparse, linear regression or data classification is necessary.	Computational speed is important, information about the matrix structure is necessary, solving a small system of linear equations is necessary.

4.3. Examples of LU and QR Factorization

Example (1): Solving a Linear System using LU Factorization

To solve the following system of linear equations using the LU factorization method:

$$2x_1 + x_2 + 3x_3 = 1$$
$$6x_1 + 2x_2 + 7x_3 = 0$$
$$4x_1 + 8x_2 + 2x_3 = 2$$

Solution:

This system can be represented in the matrix form $A\vec{x} = \vec{b}$, where:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 2 & 7 \\ 4 & 8 & 2 \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Step 1: Factorize Matrix A into L and U

Our goal is to factorize matrix A into the product of a **Lower** triangular matrix L and an **Upper** triangular matrix U, such that $A = LU$.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Applying Doolittle's Algorithm:

1. **First row of U:**

$$u_{11} = a_{11} = 2$$
$$u_{12} = a_{12} = 1$$
$$u_{13} = a_{13} = 3$$

2. **First column of L:**

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{6}{2} = 3$$
$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{4}{2} = 2$$

3. **Remaining elements of U and L:**

$$u_{22} = a_{22} - l_{21}u_{12} = 2 - (3)(1) = -1$$
$$u_{23} = a_{23} - l_{21}u_{13} = 7 - (3)(3) = -2$$
$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{8 - (2)(1)}{-1} = \frac{6}{-1} = -6$$
$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 2 - (2)(3) - (-6)(-2) = 2 - 6 - 12 = -16$$

Thus, the matrices L and U are:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -6 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -16 \end{pmatrix}$$

Verification:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -6 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -16 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 2 & 7 \\ 4 & 8 & 2 \end{pmatrix} = A$$

Step 2: Solve the Linear System

Since $A\vec{x} = \vec{b}$ and $A = LU$, then $LU\vec{x} = \vec{b}$. We can solve this system in two stages by defining an intermediate vector $\vec{y} = U\vec{x}$.

1. **Stage 1: Solve the system $L\vec{y} = \vec{b}$ to find $\vec{y}$ (Forward Substitution):**

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -6 & 1 \end{pmatrix} . \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

   - From the first row: $y_1 = 1$
   - From the second row:

$$3y_1 + y_2 = 0 \iff 3(1) + y_2 = 0 \iff y_2 = -3$$

   - From the third row:

$$2y_1 - 6y_2 + y_3 = 2 \iff 2(1) - 6(-3) + y_3 = 2 \iff 2 + 18 + y_3 = 2 \iff y_3 = -18$$

Thus, the intermediate vector is $\vec{y} = \begin{pmatrix} 1 \\ -3 \\ -18 \end{pmatrix}$.

2. **Stage 2: Solve the system $U\vec{x} = \vec{y}$ to find $\vec{x}$ (Back Substitution):**

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -16 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -18 \end{pmatrix}$$

- From the third row: $-16x_3 = -18 \Leftrightarrow x_3 = \frac{18}{16} = \frac{9}{8} = 1.125$
- From the second row:
  $$-x_2 - 2x_3 = -3 \Leftrightarrow -x_2 - 2(1.125) = -3 \Leftrightarrow -x_2 - 2.25 = -3 \Leftrightarrow -x_2 = -0.75 \Leftrightarrow x_2 = 0.75$$
- From the third row:
  $$2x_1 + x_2 + 3x_3 = 1 \Leftrightarrow 2x_1 + (0.75) + 3(1.125) = 1$$
  $$\Leftrightarrow 2x_1 + 0.75 + 3.375 = 1 \Leftrightarrow 2x_1 + 4.125 = 1 \Rightarrow 2x_1 = -3.125 \Leftrightarrow x_1 = -1.5625$$

Final Result:

The solution to the system of linear equations is:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1.5625 \\ 0.75 \\ 1.125 \end{pmatrix}$$

Thus, we have factorized matrix A into the product of a **lower triangular matrix L** and an **upper triangular matrix U**, and subsequently used this factorization to efficiently solve the linear system.

Example (2): Solving a Linear System using QR Factorization (Corrected Formulation)

To solve the following linear system (least-squares problem) using the QR factorization method:

$$\begin{pmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

This system can be represented in the matrix form $A\vec{x} = \vec{b}$.

Solution:

Step 1: Factorize Matrix A into Q and R using the Gram-Schmidt Process

Our goal is to factorize matrix A into the product of a matrix Q with orthonormal columns and an upper triangular matrix R.

1. **Define the column vectors of matrix A:**

$$\overrightarrow{u_1} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \overrightarrow{u_2} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

2. **Apply the Gram-Schmidt process:**

- **First vector:**

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

- **Second vector:**

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \text{proj}_{\overrightarrow{v_1}}(\overrightarrow{u_2}) = \overrightarrow{u_2} - \frac{\overrightarrow{u_2} \cdot \overrightarrow{v_1}}{\|\overrightarrow{v_1}\|^2}\overrightarrow{v_1}\overrightarrow{v_2}$$

$$= \begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{(3)(2)+(4)(2)+(1)(1)}{2^2+2^2+1^2}\begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{15}{9}\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{5}{3}\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\\ \frac{2}{3}\\ -\frac{2}{3}\end{pmatrix}$$

3. **Normalize the vectors to obtain the columns of Q:**
   - **First column q_1:**

$$\|\overrightarrow{v_1}\| = \sqrt{9} = 3 \implies \overrightarrow{q_1} = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{pmatrix}$$

   - **Second column q_2:**

$$\|\overrightarrow{v_2}\| = \sqrt{(-1/3)^2 + (2/3)^2 + (-2/3)^2} = \sqrt{1/9 + 4/9 + 4/9}$$

$$= \sqrt{9/9} = 1 \implies \overrightarrow{q_2} = \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|} = \begin{pmatrix}-\frac{1}{3}\\ \frac{2}{3}\\ -\frac{2}{3}\end{pmatrix}$$

4. **Construct Matrices Q and R:**

   The matrix Q is composed of the orthonormal columns:

$$Q = [\overrightarrow{q_1}, \overrightarrow{q_2}] = \begin{pmatrix}\frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3}\end{pmatrix}$$

   The matrix R is calculated from the relation $R = Q^T A$:

$$R = \begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\end{pmatrix} \cdot \begin{pmatrix}2 & 3\\ 2 & 4\\ 1 & 1\end{pmatrix} = \begin{pmatrix}3 & 5\\ 0 & 1\end{pmatrix}$$

Step 2: Solve the Linear System

The least-squares system $A\vec{x} = \vec{b}$ is solved by transforming it into the system $R\vec{x} = Q^T\vec{b}$.

1. **Calculate the right-hand side $Q^T\vec{b}$:**

$$Q^T\vec{b} = \begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\end{pmatrix} \cdot \begin{pmatrix}7\\3\\1\end{pmatrix} = \begin{pmatrix}\frac{14+6+1}{3}\\ \frac{-7+6-2}{3}\end{pmatrix} = \begin{pmatrix}7\\-1\end{pmatrix}$$

2. **Solve the system $R\vec{x} = Q^T\vec{b}$ (Back Substitution):**

$$\begin{pmatrix}3 & 5\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}7\\-1\end{pmatrix}$$

   * From the second row: $x_2 = -1$
   * From the first row: $3x_1 + 5x_2 = 7 \iff x_1 = 4$

Final Result:
The least-squares solution is:

$$\vec{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

4.4. QR Decomposition with Rational Elements

As a research contribution, we seek to determine the conditions on matrices of order two and three such that they can be decomposed into QR with rational elements.

Theorem 3: 2×2 Matrices

A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$ can be decomposed into $Q \cdot R$ with rational elements if and only if the following condition is satisfied:

$$\sqrt{a^2 + c^2} \in \mathbb{Q}$$

Proof:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$, and let us find Q in the decomposition of A into $Q \cdot R$ using the Gram-Schmidt process.
Take the two vectors:

$$v_1 = \begin{pmatrix} a \\ c \end{pmatrix}, v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$$

Set: $u_1 = v_1$
Then:

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{a^2 + c^2}} \cdot \begin{pmatrix} a \\ c \end{pmatrix}$$

Set: $u_2 = v_2 - (v_2 \cdot q_1)q_1$
Then:

$$u_2 = \begin{pmatrix} b \\ d \end{pmatrix} - \left(\frac{1}{\sqrt{a^2 + c^2}} \cdot (b\ d) \cdot \begin{pmatrix} a \\ c \end{pmatrix} \right) \cdot \left(\frac{1}{\sqrt{a^2 + c^2}} \cdot \begin{pmatrix} a \\ c \end{pmatrix} \right)$$

$$= \begin{pmatrix} b \\ d \end{pmatrix} - \frac{a \cdot b + c \cdot d}{a^2 + c^2} \cdot \begin{pmatrix} a \\ c \end{pmatrix}$$

$$= \frac{1}{a^2 + c^2} \begin{pmatrix} b(a^2 + c^2) - a(ab + cd) \\ d(a^2 + c^2) - c(ab + cd) \end{pmatrix}$$

$$= \frac{1}{a^2 + c^2} \begin{pmatrix} ba^2 + bc^2 - ba^2 - acd \\ da^2 + dc^2 - acb - dc^2 \end{pmatrix}$$

$$= \frac{1}{a^2 + c^2} \begin{pmatrix} bc^2 - acd \\ da^2 - acb \end{pmatrix}$$

$$= \frac{1}{a^2 + c^2} \begin{pmatrix} c(bc - ad) \\ a(da - cb) \end{pmatrix} = \frac{ad - bc}{a^2 + c^2} \begin{pmatrix} -c \\ a \end{pmatrix}$$

Therefore:

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{\frac{ad - bc}{a^2 + c^2}}{\frac{|ad - bc|}{a^2 + c^2} \sqrt{a^2 + c^2}} \cdot \begin{pmatrix} -c \\ a \end{pmatrix}$$

$$= \frac{\text{sign}(ad - bc)}{\sqrt{a^2 + c^2}} \cdot \begin{pmatrix} -c \\ a \end{pmatrix}$$

Finally, the matrix Q is:

$$Q = \frac{1}{\sqrt{a^2 + c^2}} \cdot \begin{pmatrix} a & -Dc \\ c & Da \end{pmatrix};\ D = \text{sign}(ad - bc)$$

Example:

For the matrix $A = \begin{pmatrix} 3 & \frac{1}{2} \\ 4 & \frac{3}{4} \end{pmatrix}$, we note that:

$$\sqrt{3^2 + 4^2} = 5$$

Therefore, the matrix Q resulting from the decomposition of A into $Q \cdot R$ will have rational elements.
Since:

$$3 \times \frac{3}{4} - 4 \times \frac{1}{2} = \frac{9}{4} - 2 = \frac{1}{4} > 0$$

We have:

$$Q = \frac{1}{5} \cdot \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Theorem 4: 3×3 Matrices
Let $A \in M_3(\mathbb{Q})$ such that:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (v_1\ v_2\ v_3)$$

The matrix Q resulting from the decomposition of A into $Q \cdot R$ by the Gram-Schmidt process has rational elements if and only if the following two conditions are satisfied:

- $\|v_1\| \in \mathbb{Q}$
- $\|v_1 \times v_2\| \in \mathbb{Q}$

Proof:
Take the vectors:

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

We follow the procedure for finding the matrix Q resulting from the decomposition of A into $Q \cdot R$ according to Gram-Schmidt.

Step 1:
Set $u_1 = v_1$
Then:

$$q_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{a_{11}^2 + a_{21}^2 + a_{31}^2}} \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \frac{1}{\sqrt{N_1}} \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

where $N_1 = \|u_1\|^2$

Step 2:
Set: $u_2 = v_2 - (v_2 \cdot q_1) \cdot q_1$
Therefore:

$$u_2 = v_2 - \frac{v_2 \cdot v_1}{N_1} \cdot v_1$$

And:

$$N_2 = \|u_2\|^2 = u_2 \cdot u_2 = \left(v_2 - \frac{v_2 \cdot v_1}{N_1} \cdot v_1\right) \cdot \left(v_2 - \frac{v_2 \cdot v_1}{N_1} \cdot v_1\right)$$

$$= \|v_2\|^2 - \frac{v_1 v_2}{N_1} v_1 \cdot v_2 - \frac{v_1 v_2}{N_1} v_2 \cdot v_1 + \frac{(v_1 v_2)^2}{N_1^2} \|v_1\|^2$$

$$= \|v_2\|^2 - 2\frac{(v_1 v_2)^2}{N_1} + \frac{(v_1 v_2)^2}{N_1^2} N_1$$

$$= \|v_2\|^2 - \frac{(v_1 v_2)^2}{N_1} = \frac{N_1 \cdot \|v_2\|^2 - (v_1 v_2)^2}{N_1}$$

By calculation, we find:

$$N_2 = \frac{\|v_1 \times v_2\|^2}{N_1}$$

Therefore:

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{v_2 - \frac{v_2 \cdot v_1}{N_1} \cdot v_1}{\frac{\|v_1 \times v_2\|}{\sqrt{N_1}}} = \frac{N_1 \cdot v_2 - (v_2 \cdot v_1) \cdot v_1}{\sqrt{N_1} \cdot \|v_1 \times v_2\|}$$

Step 3:

Set $u_3 = v_3 - (v_3 \cdot q_1) \cdot q_1 - (v_3 \cdot q_2) \cdot q_2$

Let $n = \frac{v_1 \times v_2}{\|v_1 \times v_2\|}$. This vector is orthogonal to both q_1 and q_2.

Therefore, the vector u_3 is the projection of v_3 onto n, that is:

$$u_3 = (v_3 \cdot n) \cdot n$$

Therefore:

$$\|u_3\| = |v_3 \cdot n|$$

We have:

$$v_3 \cdot n = v_3 \cdot \frac{v_1 \times v_2}{\|v_1 \times v_2\|} = \frac{v_3 \cdot (v_1 \times v_2)}{\|v_1 \times v_2\|} = \frac{\det(A)}{\|v_1 \times v_2\|}$$

Therefore:

$$\|u_3\| = \frac{|\det(A)|}{\|v_1 \times v_2\|}$$

And thus:

$$q_3 = \frac{v_3 - (v_3 \cdot q_1) \cdot q_1 - (v_3 \cdot q_2) \cdot q_2}{\frac{|\det(A)|}{\|v_1 \times v_2\|}}$$

From the above, we find the following:

- For q_1 to be rational, we must have $\sqrt{N_1} = \|v_1\| \in \mathbb{Q}$
- With the previous condition satisfied, u_2 becomes rational by its construction, and for q_2 to be rational, we must have:

$$\|u_2\| = \sqrt{N_2} = \frac{\|v_1 \times v_2\|}{\sqrt{N_1}} \in \mathbb{Q}$$

This is satisfied if: $\|v_1 \times v_2\| \in \mathbb{Q}$

- With the previous conditions satisfied, we find that u_3 is rational by its construction, and also $\|u_3\| \in \mathbb{Q}$

Thus, the desired result is proven.

Example:

For the matrix:

$$A = \begin{pmatrix} 33 & -8 & 41 \\ -6 & -19 & -62 \\ -30 & -50 & 35 \end{pmatrix}$$

which satisfies the conditions of the theorem, the matrix Q resulting from the decomposition of A into QR by the Gram-Schmidt process is:

$$Q = \begin{pmatrix} \frac{11}{15} & -\frac{2}{3} & \frac{2}{15} \\ -\frac{2}{15} & -\frac{1}{3} & -\frac{14}{15} \\ -\frac{10}{15} & -\frac{2}{3} & \frac{5}{15} \end{pmatrix}$$

5. Results

All matrix factorization methods of were reviewed, and the advantages and disadvantages of each method were examined. In addition, the principle definitions related to matrices and matrix factorization were presented. The study also established the necessary conditions for square matrices of orders two and three to produce orthonormal matrices when applying the Gram-Schmidt method for QR matrix decomposition.

6. Proposals and Recommendations

The study of matrix factorization, together with the review and comparison of previously developed factorization methods provides a foundation for developing more efficient and reliable techniques. Furthermore, identifying the limitations and drawbacks of the existing methods may contribute to the development of improved approaches that reduce computational complexity and overcome the shortcomings of earlier methods.

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