JWPU
Journal of Al-Wataniya Private University

ISSN: 3105-2150

Volume (2) Number (2)
Available at: https://doi.org/ 10.5281/zenodo.19883910

APPROXIMATION OF PERIODIC FUNCTIONS BY FEJER SUM AND DE LA VALLEE POUSSIN SUMS

Dr. Mikhael Shahoud [1,*]

ABSTRACT

In this paper, we establish several results concerning the approximation of periodic functions by Fejér and de la Vallée Poussin means in Lebesgue spaces L_2π^p.The obtained estimates are expressed in terms of function for L_2 and the second-order modulus of continuity.

The approximation of periodic functions by trigonometric polynomials plays a central role in Fourier analysis. Among the classical summation methods, Fejér sums and de la Vallée-Poussin sums provide powerful tools for improving the convergence behavior of Fourier series. In this work, we investigate the approximation of -periodic functions in spaces L_2by these two summation methods. Special attention is given to the relationship between the smoothness of the function, measured via the second-order modulus of continuity, and the rate of approximation. Our results contribute to a clearer understanding of how summability methods refine Fourier approximation and provide effective tools for both theoretical and applied analysis.

KEYWORDS: Jackson inequality; approximation; Fejér means, De La Vallee Poussin means

Submitted on September 23, 2024; Revised on November 9, 2024; Accepted on November 26, 2024

1 Faculty, Al-Wataniya Private University, Hama, Syria.

* Corresponding author. E-mail address: mikhaelshahoud@wpu.edu.sy

1. Introduction

The Hilbert space L_2 consists of all 2π- periodic functions that are second power is Lebesgue integrable functions f defined on the real line R with the norm

‖F‖_2= (1/2Π ∫_(-Π) ^Π▒|F(X)^2 DX|)(2/1)^

Consider its trigonometric Fourier series for function f(x)

A_0/2+∑_(K=1)^∞▒ 〖A_K COS⁡KX+B_K SIN⁡KX〗 (1)

and S_n be the nth partial sum of Fourier series

S_N=A_0/2+∑_(K=1)^N▒ 〖P_K COS⁡(KX+Φ_K) 〗 (2)

The Fejér sum is one of the classical arithmetic means used to improve the convergence of a Fourier series.

Σ_(N-1) (F)=(S_0+S_1+...+S_(N-1))/N

Remark (1):
For every $f(x) \in L_2$, the series $\sum_{k=1}^{\infty} {\rho_k}^2$ will be convergent (see[4])
Moreover, the terms of this series are non-negative real numbers, that become non-increase from certain index $k = n_0$
Therefore

$$\boldsymbol{{\rho_p}^2 \leq {\rho_q}^2 \qquad ,\forall p(p > q \geq n_0)}$$

We obtain

$$\boldsymbol{\frac{{\rho_p}^2}{p} \leq \frac{{\rho_q}^2}{q} \forall p, (p > q \geq n_0)}$$

Lemma (1)

Fejér sums can be written in the form of a trigonometric polynomial of degree no greater than (n-1).
Proof:
We have from equation (2):

$$\boldsymbol{S_0 = \frac{a_0}{2}}$$

$$\boldsymbol{S_1 = \frac{a_0}{2} + \rho_1 \cos(x + \phi_1)}$$

$$\boldsymbol{S_2 = \frac{a_0}{2} + \rho_1 \cos(x + \phi_1) + \rho_2 \cos(2x + \phi_2)}$$

$$\boldsymbol{S_{n-1} = \frac{a_0}{2} + \rho_1 \cos(x + \phi_1) + \rho_2 \cos(2x + \phi_2) +\ldots+ \rho_{n-1} \cos\big((n-1)x + \phi_{n-1}\big)}$$

It is obtained by summing the first n partial sums of the Fourier series and dividing by n : (see [5])

$$\boldsymbol{\sigma_{n-1}(f) = \frac{S_0 + S_1 +\ldots+ S_{n-1}}{n}}$$

$$\Rightarrow \boldsymbol{\sigma_{n-1}(f) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k \cos(kx+\phi_k)}$$

Remark (2):
From Fourier series for function (1):

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos k\,x + \beta_k \sin k\,x$$

Therefore

$$f(x) - \sigma_{n-1}(f) = \sum_{k=1}^{n-1} \frac{k}{n}\rho_k \cos(kx+\phi_k) + \sum_{k=n}^{\infty} \rho_k \cos(kx+\phi_k)$$

From Parseval equality

$$\|f(x) - \sigma_{n-1}(f)\|^2 = \sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2 \rho_k^2 + \sum_{k=n}^{\infty}\rho_k^2 \; (3).$$

And De La Vallee Poussin sum is one of the arithmetic means of the partial sums of a Fourier series from mth term to (n-1)th term .

$$V_m^{n-1}(f) = \frac{S_m + S_{m+1} + \ldots\ldots\ldots + S_{n-1}}{n-m}$$

Lemma (2):
De La Vallee Poussin sums can be written in the form of a trigonometric polynomial of degree no greater than (n-1).(see[3]and [6])
Proof:
We have from equation (2):

$$S_m = \frac{a_0}{2} + \sum_{k=1}^{m} \rho_k \cos(kx+\phi_k)$$

$$S_{m+1} = S_m + \rho_{m+1}\cos[(m+1)x + \phi_{m+1}]$$

$$S_{m+2} = S_m + \rho_{m+1}\cos[(m+1)x + \phi_{m+1}] + \rho_{m+2}\cos[(m+2)x + \phi_{m+2}]$$

.

.

$$S_{n-1} = S_m + \rho_{m+1}\cos[(m+1)x + \phi_{m+1}] + \rho_{m+2}\cos[(m+2)x + \phi_{m+2}] + \ldots\ldots + \\ + \rho_{n-1}\cos[(n-1)x + \phi_{n-1}]$$

By sum and dived by m-n:

$$V_m^{n-1}(f) = S_m + \sum_{k=m+1}^{n-1}\left(1 - \frac{k-m}{n-m}\right)\rho_k \cos(kx+\phi_k)$$

Remark (3):
From Fourier series for function:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos k\,x + \beta_k \sin k\,x$$

Therefore

$$\boldsymbol{f(x) - V_m^{n-1}(f) = \sum_{k=m+1}^{n-1}\left(\frac{k-m}{n-m}\right)\rho_k \cos(kx+\phi_k) + \sum_{k=n}^{\infty}\rho_k \cos(kx+\phi_k)}$$

From Parseval equality

$$\|f(x) - V_m^{n-1}(f)\|^2 = \sum_{k=m+1}^{n-1}\left(\frac{k-m}{n-m}\right)^2 \rho_k^2 + \sum_{k=n}^{\infty}\rho_k^2 \ (4).$$

In [8] , if $f \in lip(\delta^{\alpha}, C)$, then

$$|V_n^1 f(x) - f(x)| << \begin{cases} \frac{1}{(\lambda_n+1)^{\alpha}} & when \quad 0<\alpha<1 \\ \frac{1+ln(\lambda_n+1)}{(\lambda_n+1)} & when \quad \alpha = 1 \end{cases}$$

holds true uniformly in x.
In [9] , if $f \in L^1$, then

$$|\sigma_n f(x) - f(x)| << \frac{1}{n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) \qquad ,(n = 0,1,\ldots\ldots)$$

holds for all real x.
In [3] , let $f \in L^1$, If $\frac{n}{\lambda_n} = O(1)$,then

$$\left|V_n^{\gamma} f(x) - f(x)\right| << \begin{cases} \frac{1}{n+1}\sum_{k=0}^{n}\Omega_x f\left(\frac{\pi}{k+1}\right) & ,(\gamma \geq 1) \\ \frac{1}{(n+1)^{\gamma}}\sum_{k=1}^{n}\frac{1}{(k+1)^{1-\gamma}}\Omega_x f\left(\frac{\pi}{k+1}\right) & ,(0<\gamma<1) \end{cases}$$

In [2] , for each $1 < p < \infty$ there exists a constant C_p such that ; for each $f \in X^p$;

$$\frac{1}{4C_p}\|f - \sigma_n(f)\|_p \leq \omega_1\left(f, \frac{3}{n+1}\right) \leq 8C_p\|f - \sigma_n(f)\|_p$$

and in particular when $p = 2$

$$\frac{1}{4}\|f - \sigma_n(f)\|_2 \leq \omega_1\left(f, \frac{3}{n+1}\right) \leq 8\|f - \sigma_n(f)\|_2$$

In [1] , Ching and Chui proved that ; for each $f(x) \in X^2$and $n \in N$;

$$\frac{1}{\pi}\omega_2\left(\frac{\pi}{n+1}; f\right) < \|\sigma_n(f) - f\| < \frac{1}{\sqrt{2}}\omega_2\left(\frac{\pi}{n+1}; f\right)$$

In this manuscript we show that similar relation for $f(x) \in L_2$.

2. Methodology

In this research, we follow a theoretical and analytical approach based on classical approximation theory.
The aim is to study the approximation of periodic functions by Fejér and de la Vallée Poussin means, and to examine their convergence behavior and related error estimates.
The methodology includes of the following main steps:

Function Space and Assumptions**:**
We consider 2π -periodic functions belonging to the space $L^p[-\pi,\pi]$ for suitable values of P, as well as continuous functions where applicable.
Standard assumptions on the smoothness and integrability of the target function are clearly stated to ensure that the approximation results are valid and applicable.

1. **Definition of Approximation Operators:**
   The Fejér mean $\sigma_n f(x)$ and the de la Vallée Poussin mean $V_m^n(f)$ are defined using the corresponding trigonometric. We Their main properties, such as positivity and reproducing behaviour, are analyzed to support the theoretical framework of the study.
2. **Analytical Derivation:**
   Using tools from Fourier analysis, we derive upper bounds for the approximation errors. In particular, we apply estimates based on the second - order modulus of continuity and use classical inequalities (such as Jackson-type inequalities) to quantify the rate of convergence.
3. **Comparative Analysis:**
   A detailed comparison between the Fejér and de la Vallée Poussin sums is conducted based on their approximation performance. We highlight conditions under which one method may provide a better rate of convergence or smoother approximation behavior.

This methodology ensures a rigorous and structured analysis of the approximation capabilities of both Fejér and de la Vallée Poussin sums within the framework of harmonic analysis and functional approximation.

3. Results and discussion

Theorem (1): for any function$f(x) \in L_2, (f(x) \neq const)$and for any natural number n the following inequality holds:

$$\|f(x) - \sigma_{n-1}(f)\| \leq \frac{1}{\sqrt{6}}\left(\frac{n}{\pi}\int_0^{\frac{\pi}{n}} \omega_2^2(f,t)dt\right)^{\frac{1}{2}} \quad , \quad \forall n \geq n_0$$

Proof:
From Fourier series and Parseval equality will be obtained:

$$\|f(x-t) - 2f(x) + f(x+t)\|^2 = 4\sum_{k=1}^{\infty} {\rho_k}^2 (1 - \cos k\, t)^2$$

On the other hand, using the definition of second-order modulus of continuity [7] we have:

$$\omega_2(f,\delta) \geq \|f(x-\delta) - 2f(x) + f(x+\delta)\|$$

$$\omega_2{}^2(f,\delta) \geq \|f(x-\delta) - 2f(x) + f(x+\delta)\|^2 \geq$$

$$\geq 4\sum_{k=1}^{\infty} {\rho_k}^2 (1 - \cos k\, \delta)^2$$

$$\geq 4\sum_{k=1}^{\infty}\rho_k{}^2(1-\cos k\,\delta)^2 \geq 4\sum_{k=1}^{n-1}\rho_k{}^2(1-\cos k\,\delta)^2 + 4\sum_{k=n}^{\infty}\rho_k{}^2(1-\cos k\,\delta)^2$$

$$\geq 4\sum_{k=1}^{n-1}\rho_k{}^2(1-\cos k\,\delta)^2 + 2\sum_{k=n}^{\infty}\rho_k{}^2(3-4\cos k\,\delta+\cos 2\,k\delta)$$

$$\frac{1}{6}\omega_2{}^2(f,\delta) \geq \frac{2}{3}\sum_{k=1}^{n-1}\rho_k{}^2(1-\cos k\,\delta)^2 + \sum_{k=n}^{\infty}\rho_k{}^2 - \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,\delta + \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,k\delta$$

Changingδ to t we can write

$$\sum_{k=n}^{\infty}\rho_k{}^2 \leq \frac{1}{6}\omega_2{}^2(f,t) - \frac{2}{3}\sum_{k=1}^{n-1}\rho_k{}^2(1-\cos k\,t)^2 + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

Summing both sides by $\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2\rho_k^2$we obtain:

$$\sum_{k=n}^{\infty}\rho_k{}^2 + \sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2\rho_k^2 \leq \frac{1}{6}\omega_2{}^2(f,t) + \sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^2\rho_k^2 - \frac{2}{3}\sum_{k=1}^{n-1}\rho_k{}^2(1-\cos k\,t)^2 + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

from (3) we obtain

$$\|f(x)-\sigma_{n-1}(f)\|^2 \leq \frac{1}{6}\omega_2{}^2(f,t) + \sum_{k=1}^{n-1}\rho_k^2\left[\left(\frac{k}{n}\right)^2 - \frac{2}{3}(1-\cos k\,t)^2\right] + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

IN PARTICULAR $\sum_{k=1}^{n-1}\rho_k^2\left(\frac{k}{n}\right)^2 \leq \sum_{k=1}^{n-1}\rho_k^2\sin^2\left(\frac{kt}{2}\right)$ *(SEE [1])*

and hence

$$\|f(x)-\sigma_{n-1}(f)\| \leq \frac{1}{6}\omega_2{}^2(f,t) + \sum_{k=1}^{n-1}\rho_k^2\left[\sin^2\left(\frac{kt}{2}\right) - \frac{2}{3}(1-\cos k\,t)^2\right] + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

$$\leq \frac{1}{6}\omega_2{}^2(f,t) + \sum_{k=1}^{n-1}\rho_k^2\left[\frac{1}{2}(1-\cos k\,t) - \frac{2}{3}(1-\cos k\,t)^2\right] + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

$$\leq \frac{1}{6}\omega_2{}^2(f,t) + \sum_{k=1}^{n-1}\rho_k^2\left[(1-\cos k\,t)\left(\frac{1}{2} - \frac{2}{3}(1-\cos k\,t)\right)\right] + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

$$\leq \frac{1}{6}\omega_2{}^2(f,t) + \sum_{k=1}^{n-1}\rho_k^2\left[(1-\cos k\,t)\left(-\frac{1}{6} + \frac{2}{3}\cos k\,t\right)\right] + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

$$\leq \frac{1}{6}\omega_2{}^2(f,t) + \frac{4}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos k\,t - \frac{1}{3}\sum_{k=n}^{\infty}\rho_k{}^2\cos 2\,kt$$

By integrating with respect to t form 0 to $\frac{\pi}{n}$ we obtain

$$\frac{\pi}{n}\|f(x)-\sigma_{n-1}(f)\|^2 \leq \frac{1}{6}\int_0^{\frac{\pi}{n}}\omega_2{}^2(f,t)dt + \frac{4}{3}\sum_{k=n}^{\infty}\frac{\rho_k{}^2}{k}\sin\frac{k\pi}{n} - \frac{1}{6}\sum_{k=n}^{\infty}\frac{\rho_k{}^2}{k}\sin\frac{2k\pi}{n}$$

Let's prove that

$\sum_{k=n}^{\infty}\frac{\rho_k{}^2}{k}\sin\left(\frac{k\pi}{n}\right) \leq 0$ (see Appendix A)

$\sum_{k=n}^{\infty}\frac{\rho_k{}^2}{k}\sin\frac{2k\pi}{n} \geq 0$ (see Appendix B)

Therefore

$$\frac{\pi}{n}\|f(x)-\sigma_{n-1}(f)\|^2 \leq \frac{1}{6}\int_0^{\frac{\pi}{n}}\omega_2{}^2(f,t)dt$$

$$\Rightarrow \|f(x)-\sigma_{n-1}(f)\| \leq \frac{1}{\sqrt{6}}\left\{\frac{n}{\pi}\int_0^{\frac{\pi}{n}}\omega_2{}^2(f,t)dt\right\}^{\frac{1}{2}}$$

Theorem (2): for any function $f(x) \in L_2, (f(x) \neq const)$ and for any natural number $m < n; n, m$ the following inequality holds:

$$\|f(x) - V_m^{n-1}(f)\| \leq \frac{1}{\sqrt{6}}\left(\frac{n}{\pi}\int_0^{\frac{\pi}{n}}\omega_2^2(f,t)dt\right)^{\frac{1}{2}} \quad , \quad \forall n \geq n_0$$

Proof:

To prove the inequality, it's enough to prove

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{V}_{\boldsymbol{m}}^{\boldsymbol{n-1}}(\boldsymbol{f})\| \leq \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{\sigma}_{\boldsymbol{n-1}}(\boldsymbol{f})\|$$

We have

$$\frac{\boldsymbol{k}-\boldsymbol{m}}{\boldsymbol{n}-\boldsymbol{m}} \leq \frac{\boldsymbol{k}}{\boldsymbol{n}}$$

THEREFORE

$$\sum_{k=m+1}^{n-1}\left(\frac{k-m}{n-m}\right)^2\rho^2\le\sum_{k=m+1}^{n-1}\left(\frac{k}{n}\right)^2\rho^2$$

THEREFORE

$$\sum_{k=m+1}^{n-1}\left(\frac{k-m}{n-m}\right)^2\rho^2+\sum_{k=n}^{\infty}\rho_k^2\le\sum_{k=m+1}^{n-1}\left(\frac{k}{n}\right)^2\rho^2+\sum_{k=n}^{\infty}\rho_k^2$$

$$\|f(x)-V_m^{n-1}(f)\|^2=\sum_{k=m+1}^{n-1}\left(\frac{k-m}{n-m}\right)^2\rho^2+\sum_{k=n}^{\infty}\rho_k^2\le\sum_{k=m+1}^{n-1}\left(\frac{k}{n}\right)^2\rho^2+\sum_{k=n}^{\infty}\rho_k^2$$
$$=\|f(x)-\sigma_{n-1}(f)\|^2$$

3.1 Appendix

A. We now proceed to prove the following result.

$$\sum_{k=n}^{\infty}\frac{{\rho_k}^2}{k}sin\left(\frac{k\pi}{n}\right)\le 0$$

We collect the terms of this series in the following form

$$\sum_{k=n}^{\infty}\frac{{\rho_k}^2}{k}sin\left(\frac{k\pi}{n}\right)=$$
$$\sum_{l=1}^{\infty}\sum_{j=0}^{n-1}\left\{\frac{{\rho^2}_{(2l-1)n+j}}{(2l-1)n+j}sin[(2l-1)n+j]\frac{\pi}{n}+\frac{{\rho^2}_{2nl+j}}{2nl+j}sin[2nl+j]\frac{\pi}{n}\right\}$$

On the other hand

$$sin[(2l-1)n+j]\frac{\pi}{n}=sin\left(2l\pi-\pi+\frac{j\pi}{n}\right)=sin\left(\frac{j\pi}{n}-\pi\right)=-sin\left(\frac{j\pi}{n}\right),$$
$$sin[2nl+j]\frac{\pi}{n}=sin\left(2l\pi+\frac{j\pi}{n}\right)=sin\left(\frac{j\pi}{n}\right)$$

We obtain

$$\sum_{k=n}^{\infty}\frac{\rho^2}{k}sin\left(\frac{k\pi}{n}\right)=\sum_{l=1}^{\infty}\sum_{j=0}^{n-1}\left[\frac{{\rho^2}_{2nl+j}}{2nl+j}-\frac{{\rho^2}_{(2l-1)n+j}}{(2l-1)n+j}\right]sin\left(\frac{j\pi}{n}\right)$$

We have

$$\forall l\ge 1\Rightarrow 2nl+j>(2l-1)n+j$$

From remark (1)

$$\frac{\rho^2_{2nl+j}}{2nl+j}\le\frac{\rho^2_{(2l-1)n+j}}{(2l-1)n+j}\qquad\forall l\ge 1\quad,\forall j\ge 0$$

And also

$$\sin\left(\frac{j\pi}{n}\right) \geq 0 \qquad , \quad 0 \leq j \leq n-1$$

Therefore

$$\sum_{l=1}^{\infty}\sum_{j=0}^{n-1}\left[\frac{{\rho^2}_{2nl+j}}{2nl+j} - \frac{{\rho^2}_{(2l-1)n+j}}{(2l-1)n+j}\right]\sin\left(\frac{j\pi}{n}\right) \leq 0$$

$$\Rightarrow \sum_{k=n}^{\infty}\frac{\rho^2}{k}\sin\left(\frac{k\pi}{n}\right) \leq 0$$

B. We now proceed to prove the following result.

$$\sum_{k=n}^{\infty}\frac{{\rho_k}^2}{k}\sin\frac{2k\pi}{n} \geq 0$$

To prove this inequality, we distinguish between two cases

First case: Let n be an odd number. Then, we use the symbol $[\frac{n}{2}]$ to denote the integer part of a number $\frac{n}{2}$.

We collect the terms of this series in the following form

$$\sum_{k=n}^{\infty}\frac{{\rho_k}^2}{k}\sin\frac{2k\pi}{n} = \sum_{l=1}^{\infty}\left(\sum_{j=0}^{[\frac{n}{2}]}\frac{{\rho^2}_{ln+j}}{ln+j}\sin 2\,(ln+j)\frac{\pi}{n} + \sum_{j=1}^{[\frac{n}{2}]}\frac{{\rho^2}_{(l+1)n-j}}{(l+1)n-j}\sin 2\,[(l+1)n-j]\frac{\pi}{n}\right)$$

$$= \sum_{l=1}^{\infty}\left(\sum_{j=0}^{[\frac{n}{2}]}\frac{{\rho^2}_{ln+j}}{ln+j}\sin\frac{2j\pi}{n} - \sum_{j=1}^{[\frac{n}{2}]}\frac{{\rho^2}_{(l+1)n-j}}{(l+1)n-j}\sin\frac{2j\pi}{n}\right)$$

When $j = 0$ we obtain $\sin\frac{2j\pi}{n} = 0$ therefore

$$\sum_{k=n}^{\infty}\frac{{\rho^2}_k}{k}\sin\frac{2k\pi}{n} = \sum_{l=1}^{\infty}\sum_{j=1}^{[\frac{n}{2}]}\left(\frac{{\rho^2}_{ln+j}}{ln+j} - \frac{{\rho^2}_{(l+1)n-j}}{(l+1)n-j}\right)\sin\frac{2j\pi}{n}$$

When $1 \leq j \leq [\frac{n}{2}]$ we obtain $\sin\frac{2j\pi}{n} > 0$.

and $\forall l \geq 1 \, ln + j < (l+1)n - j$

From remark (1)

$$\frac{{\rho^2}_{ln+j}}{ln+j} \geq \frac{{\rho^2}_{(l+1)n-j}}{(l+1)n-j}$$

Therefore

$$\sum_{l=1}^{\infty}\sum_{j=1}^{[\frac{n}{2}]}\left(\frac{\rho^2{}_{ln+j}}{ln+j}-\frac{\rho^2{}_{(l+1)n-j}}{(l+1)n-j}\right)sin\frac{2j\pi}{n}\geq 0$$

$$\Rightarrow\sum_{k=n}^{\infty}\frac{\rho^2{}_k}{k}sin\frac{2k\pi}{n}\geq 0$$

Second case: Let n be an even number

We collect the terms of this series in the following form

$$\sum_{k=n}^{\infty}\frac{\rho_k{}^2}{k}sin\frac{2k\pi}{n}=\sum_{l=1}^{\infty}\left(\sum_{j=0}^{\frac{n}{2}-1}\frac{\rho^2{}_{ln+j}}{ln+j}sin\,2\,(ln+j)\frac{\pi}{n}+\sum_{j=1}^{\frac{n}{2}}\frac{\rho^2{}_{(l+1)n-j}}{(l+1)n-j}sin\,2\,[(l+1)-j]\frac{\pi}{n}\right)$$

$$=\sum_{l=1}^{\infty}\left(\sum_{j=0}^{\frac{n}{2}-1}\frac{\rho^2{}_{ln+j}}{ln+j}sin\frac{2j\pi}{n}-\sum_{j=1}^{\frac{n}{2}}\frac{\rho^2{}_{(l+1)n-j}}{(l+1)n-j}sin\frac{2j\pi}{n}\right)$$

When $j=0$ we obtain $sin\frac{2j\pi}{n}=0$

When $j=\frac{n}{2}$ we obtain $sin\frac{2j\pi}{n}=0$ therefore

$$\sum_{k=n}^{\infty}\frac{\rho^2{}_k}{k}sin\frac{2k\pi}{n}=\sum_{l=1}^{\infty}\sum_{j=1}^{\frac{n}{2}-1}\left(\frac{\rho^2{}_{ln+j}}{ln+j}-\frac{\rho^2{}_{(l+1)n-j}}{(l+1)n-j}\right)sin\frac{2j\pi}{n}\geq 0$$

4. Conclusion

This paper presents the approximation of periodic function using Fejér means and de la Vallee Poussin means in Lebesgue spaces.

The approach ensures the validity of Jackson's inequality by applying the FEJER and DE LA VALLEE POUSSIN means and the second-order modulus of continuity in space L_2.

The results were as follow:

Theorem (1): For any function$f(x)\in L_2,(f(x)\neq const)$and for any natural number $n$will be obtained the inequality:

$$\|f(x)-\sigma_{n-1}(f)\|\leq\frac{1}{\sqrt{6}}\left(\frac{n}{\pi}\int_0^{\frac{\pi}{n}}\omega_2^2(f,t)dt\right)^{\frac{1}{2}}\qquad,\ \forall n\geq n_0$$

Theorem (2): For any function$f(x)\in L_2,(f(x)\neq const)$ and for any natural number $m<n;n,m$Will be obtained the inequality:

$$\|f(x)-V_m^{n-1}(f)\|\leq\frac{1}{\sqrt{6}}\left(\frac{n}{\pi}\int_0^{\frac{\pi}{n}}\omega_2^2(f,t)dt\right)^{\frac{1}{2}}\qquad,\ \forall n\geq n_0$$

References

[1] C.-H. Ching and C. K. Chui, "Some inequalities in trigonometric approximation," *Bulletin of the Australian Mathematical Society*, vol. 8, pp. 393–395, 1973, doi: 10.1017/S0004972700042684.

[2] J. Bustamante, "Direct and strong converse inequalities for approximation with Fejér means," *Demonstratio Mathematica*, vol. 53, no. 1, pp. 80–85, 2020, doi: 10.1515/dema-2020-0051.

[3] X. Z. Krasniqi, W. Łenski, and B. Szal, "Approximation of integrable functions by generalized de la Vallée Poussin means of the positive order," *Journal of Applied Analysis and Computation*, vol. 12, no. 1, pp. 106–124, Feb. 2022, doi: 10.11948/20210067.

[4] N. K. Bari, Trigonometric Series. Moscow, Russia: *Fizmatgiz*, 1961, in Russian.

[5] A. Zygmund, Trigonometric Series, 3rd ed. Cambridge, U.K.: Cambridge University Press, 2002.

[6] L. Leindler, "On summability of Fourier series," *Acta Scientiarum Mathematicarum (Szeged)*, vol. 29, pp. 147–162, 1968.

[7] Z. Ditzian and V. Totik, Moduli of Smoothness. New York, NY, USA: *Springer-Verlag*, 1987, doi: 10.1007/978-1-4612-4778-4.

[8] W. Łenski, "On the rate of pointwise summability of Fourier series," *Applied Mathematics E-Notes*, vol. 1, pp. 143–148, 2001.